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Reduction formulae for generalised hypergeometric functions of one variable

J E Gottschalk and E N Maslen

Department of Physics, University of Western Australia, Nedlands, Western Australia 6009, Australia

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Abstract. Series of gamma functions can usually be expressed as generalised hypergeometric functions of one or more variables, providing a tool for classifying series transformations. While most recent studies of these functions concern the multivariable type, the authors have derived new transformations and extended known results for single-variable functions. The derivations of the new identities are straightforward and the results are presented in their most general form.

1. Introduction

The motivation for this study of generalised hypergeometric functions is provided by the analysis of certain partial differential equations by series solution. The partial differential equations are related to the Schrödinger equation for a system of three charged particles. The solution is expressed as a series of orthogonal polynomials and powers of length variables. Recent examples are due to Pluvinage (1982), Abbott and Maslen (1987) and Gottschalk *et al* (1987). Earlier work is described by those authors. The expansion coefficients are derived in the form of single or multiple series, which require simplification. The expansion itself can be compacted, at least partially, and the technique has provided the form of the wavefunction, at small interparticle distances, in terms of special functions (Gottschalk *et al* 1987). Extension of this work involves reducing a large number of series. The generalised hypergeometric notation (Slater 1966) provides a means of classifying transformations of series, reducing the possibility of duplicating known results.

The algorithmic problem of determining indefinite integrals has been solved for an important class of integrands. The technique, described by Risch (1969, 1970) applies to integrands which are algebraic or elementary transcendental functions. This method, described by Norman (1983), provides the algorithmic basis for symbolic computation with this class of integrals. Some progress has been made in extending the Risch procedure to integrals of transcendental functions which evaluate to error functions and logarithmic integrals (Cherry and Caviness 1984, Knowles 1986).

An analogous procedure for simplifying series is needed but little progress has been made. Karr (1981) derived an algorithm which, for a limited class of finite series, produces a rational expression equal to the original series. However, Karr's algorithm does not apply to a wide variety of sums. In particular the method cannot be applied if the upper summation limit appears in the summand.

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Unlike the task of simplifying indefinite integrals, the reduction of series is generally unsystematic. Heuristic methods exist but for a given series it is not possible to state with certainty what method will succeed, or even if the series can be reduced to a specified set of functions.

Following the notation of Barnes (1907), the generalised hypergeometric function is defined by

$$ {}_pF_q\left[\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{\Pi_{i=1}^{p} (a_i)_n z^n}{\Pi_{j=1}^{q} (b_j)_n n!} \equiv {}_pF_q(z) \tag{1} $$

where $(a)_n \equiv \Gamma(a+n)/\Gamma(a)$ is the Pochhammer function. The series with $p=2$ and $q=1$ is known as the Gauss series or ordinary hypergeometric series. This function is described by Abramowitz and Stegun (1972). Bailey (1935) and Slater (1966) discuss its convergence. The parameters a_i and b_j and the variable z, which may in principle be complex, are restricted to real values in this work. Because generalisations of (1) also exist, this series is referred to here as the hypergeometric series rather than as the generalised hypergeometric series.

Calculations of the expansion coefficients for the few-particle wavefunctions frequently generate sums over polygamma and gamma functions. Many of these are nearly-poised, well-poised or Saalschutzian hypergeometric functions and hypergeometrics with numerator and denominator parameters differing by integers. Some have arguments of ± 1.

This article first describes a 'core' of hypergeometric identities, some of which are believed to be new, while others extend known results. Several methods are used in their derivation, and further extensions are suggested. Hypergeometric functions are rarely in a form in which these formulae can be applied directly. The order of the function must be reduced, as described in a later section.

The labelling of the subsections is necessarily approximate since deriving new hypergeometric identities may require integral methods which, in turn, use series rearrangement.

Writing sums as hypergeometric functions has the great advantage of simplifying manipulation by computer algebraic methods. Computer algebra is becoming more widely used and there is growing interest in encoding hypergeometric identities (Lafferty 1979, Hayden and Lamagna 1986). For this reason some known formulae are expressed here in their generalised form.

2. Elementary hypergeometric identities

The hypergeometric functions for the calculation of the few-body wavefunction form three classes:

(i) ${}_pF_{p-1}(z)$ whose top parameters and bottom parameters differ by integers,
(ii) Saalschutzian functions, and
(iii) well poised and nearly-poised series.

The terminology is that of Slater (1966).

For class (i), high-order functions are reduced to low-order functions which simplify using a core of hypergeometric identities. This reduction of order is described in § 4. Most core identities are standard reductions of hypergeometric series to gamma functions by Gauss's, Saalschutz's, Kummer's and Vandermonde's theorems (Slater 1966) and identities for ${}_2F_1$ of variable argument (Abramowitz and Stegun 1972). Other elementary identities are found by writing the ${}_pF_{p-1}$ as a series and using the polygamma ($\psi(x)$, $\psi^{(n)}(x)$), generalised Riemann zeta ($\zeta(n, x)$) and Lerch's functions ($\Phi(x, n, a)$), described by Gradshteyn and Ryzhik (1980) and Lerch (1887),

$$ {}_3F_2\left[\begin{matrix} 1, a, b \\ a+1, b+1 \end{matrix}; 1\right] = \frac{ab}{(a-b)}(\psi(a)-\psi(b)) \tag{2a} $$

$$ {}_pF_{p-1}\left[\begin{matrix} 1, a, \ldots, a \\ a+1, \ldots, a+1 \end{matrix}; z\right] = a^{p-1}\Phi(z, p-1, a) \tag{2b} $$

$$ {}_pF_{p-1}\left[\begin{matrix} 1, a, \ldots, a \\ a+1, \ldots, a+1 \end{matrix}; 1\right] = a^{p-1}\zeta(p-1, a) = \frac{(-a)^{p-1}}{(p-2)!}\psi^{(p-2)}(a) \tag{2c} $$

$$ {}_pF_{p-1}\left[\begin{matrix} 1, a, \ldots, a \\ a+1, \ldots, a+1 \end{matrix}; -1\right] = \frac{(-a/2)^{p-1}}{(p-2)!}(\psi^{(p-2)}(a/2)-\psi^{(p-2)}(a/2+\tfrac{1}{2})). \tag{2d} $$

3. Derivation of extra hypergeometric identities

The standard identities in § 2 are a subset of the core identities. Some hypergeometric functions of higher order may be related to functions in this list, achieving a complete reduction of the original series. However, the list must be expanded for work on the few-body Schrödinger equation. The expansions required are described in this section. Many generalise known results and, as the aim is to encode all the mathematics on hypergeometric functions into a computer algebra system, the most general identities are given.

It is assumed that the hypergeometric functions are convergent and do not contain negative integers in the bottom parameter list. Abramowitz and Stegun (1972) provide reductions for Gaussian hypergeometric series, ${}_2F_1(1, a; a+1; z)$ for particular values of a. These may be extended to yield

$$ {}_2F_1(1, a; a+1; z) = -az^{-a}\ln(1-z) - az^{-a}\sum_{i=0}^{a-2}\frac{z^{i+1}}{(i+1)} \tag{3a} $$

where $a \in \mathbb{Z}^+$, $|z|<1$ or $z=-1$,

$$ {}_2F_1(1, a; a+1; -z^2) = 2az^{-2a}(-1)^{a-1/2}\tan^{-1} z + 2a(-z^2)^{1/2-a}\mathop{\mathsf{S}}_{i=a-1/2}^{-1}\frac{(-z^2)^i}{(2i+1)} \tag{3b} $$

where $(2a) \in \mathbb{Z}$, but $a \notin \mathbb{Z}$, $|z| \leq 1$,

$$ {}_2F_1(1, a; a+1; z^2) = az^{-2a}\ln\left(\frac{1+z}{1-z}\right) + 2az^{-2a}\mathop{\mathsf{S}}_{i=a-1/2}^{-1}\frac{z^{2i+1}}{(2i+1)} \tag{3c} $$

where $(2a) \in \mathbb{Z}$ but $a \notin \mathbb{Z}$, $|z|<1$, and $\mathsf{S}_{r=p}^{q-1}$ is the generalised sum interpreted as $\Sigma_{r=p}^{q-1}$ for $p<q$, as $-\Sigma_{r=q}^{p-1}$ for $p>q$ and as zero when $p=q$. These may be derived from the relation between a hypergeometric and Lerch's function (2b) but are important enough to be listed separately.

3.1. *Identities obtained by differentiation*

Consider the function

$$ H = {}_{p+1}F_p\left[\begin{matrix} a, \ldots, a, b \\ a+1, \ldots, a+1 \end{matrix}; z\right]. $$

Writing part of the summand in the series representation of this function as an integral using a Laplace transform (Abramowitz and Stegun 1972) yields

$$\begin{aligned} H &= \sum_{t=0}^{\infty} \frac{(b)_t z^t a^p}{t!\Gamma(p)} \int_0^{\infty} \mathrm{e}^{-ty} y^{p-1} \mathrm{e}^{-ay}\, \mathrm{d}y \\ &= -\frac{(-a)^p}{\Gamma(p)} \int_0^1 \ln^{(p-1)}(x) x^{a-1} (1-zx)^{-b}\, \mathrm{d}x \\ &= -\frac{(-a)^p}{\Gamma(p)} \left[\frac{\mathrm{d}^{(p-1)}}{\mathrm{d}r^{(p-1)}} \int_0^1 x^{r-1}(1-zx)^{-b}\, \mathrm{d}x \right]_{r=a}. \end{aligned} \tag{4}$$

For $z=1$ this is a standard integral, giving,

$$\begin{aligned} {}_{p+1}F_p\left[\begin{matrix} a, \ldots, a, b \\ a+1, \ldots, a+1 \end{matrix}; 1\right] \\ = -\frac{\Gamma(1-b)(-a)^p}{\Gamma(p)} \left[\frac{\mathrm{d}^{(p-1)}}{\mathrm{d}r^{(p-1)}} \frac{\Gamma(r)}{\Gamma(r+1-b)} \right]_{r=a} \qquad b \notin \mathbb{Z}^+. \end{aligned} \tag{5}$$

In the case of $z \neq 1$ and $b \in \mathbb{Z}^+$ the integral is calculated by repeated integration by parts. An efficient method is based on integration by parts yielding a difference equation of the form

$$f(m) = g(m) + h(m) f(m-n)$$

with the well known solution (Milne-Thompson 1951)

$$f(m) = f(s) \prod_{i=s,n}^{m-n} h(i+n) + \sum_{i=s,n}^{m-n} g(i+n) \prod_{j=i+n,n}^{m-n} h(j+n). \tag{6}$$

This yields

$$\int x^{r-1}(1-zx)^{-b}\, \mathrm{d}x = \frac{\Gamma(b-r)}{\Gamma(b)\Gamma(1-r)} \int \frac{x^{r-1}}{(1-zx)}\, \mathrm{d}x + \frac{x^r \Gamma(b-r)}{\Gamma(b)} \sum_{i=1}^{b-1} \frac{\Gamma(i)}{(1-zx)^i \Gamma(i+1-r)}$$

where $b \in \mathbb{Z}^+$. Using $\int x^{r-1}(1-zx)^{-1}\, \mathrm{d}x = x^r \Phi(zx, 1, r)$ equation (4) becomes

$$H = -\frac{(-a)^p}{\Gamma(p)\Gamma(b)} \left[\frac{\mathrm{d}^{(p-1)}}{\mathrm{d}r^{(p-1)}} \left(\Gamma(b-r) \sum_{i=1}^{b-1} \frac{\Gamma(i)}{(1-z)^i \Gamma(i+1-r)} + \frac{\Gamma(b-r)}{\Gamma(1-r)} \Phi(z, 1, r) \right) \right]_{r=a}$$

where $b \in \mathbb{Z}^+$. In particular, for $z=-1$ and $b \in \mathbb{Z}^+$

$$\begin{aligned} {}_{p+1}F_p\left[\begin{matrix} a, \ldots, a, b \\ a+1, \ldots, a+1 \end{matrix}; -1\right] = -\frac{(-a)^p}{\Gamma(p)\Gamma(b)} \left[\frac{\mathrm{d}^{(p-1)}}{\mathrm{d}r^{(p-1)}} \left(\Gamma(b-r) \sum_{i=1}^{b-1} \frac{\Gamma(i)}{2^i \Gamma(i+1-r)} \right.\right. \\ \left.\left. + \frac{\Gamma(b-r)}{2\Gamma(1-r)} (\psi(r/2+\tfrac{1}{2}) - \psi(r/2)) \right) \right]_{r=a}. \end{aligned} \tag{7}$$

This generalises ($2d$) but the form of the previous result is more useful when $b=1$.

3.2. Identities obtained by integration

Some hypergeometric functions of variable argument reduce by relating the function to an integral of a function of lower order. The original hypergeometric is simplified if this lower-order function is reducible and the integral can be evaluated. Reduction of various ${}_2F_1(z)$ and ${}_3F_2(z)$ have been achieved using this procedure.

Expressing a hypergeometric function in terms of an integral using a Laplace transform, as described above, gives

$$
\begin{aligned}
{}_2F_1\left[\begin{matrix} a, b \\ a+1 \end{matrix}; z\right] &= \sum_{t=0}^{\infty} \frac{(b)_t z^t}{t!} a \int_0^{\infty} \mathrm{e}^{-y(t+a)}\, \mathrm{d}y \\
&= a \int_0^1 x^{a-1}{}_1F_0(b; -; zx)\, \mathrm{d}x \\
&= a \int_0^1 x^{a-1}(1-zx)^{-b}\, \mathrm{d}x.
\end{aligned}
$$

Integration by parts and using (6) relates this to standard integrals in the cases of $a \in \mathbb{Z}^+$, $(2a) \in \mathbb{Z}$ with $b \in \mathbb{Z}^+$ and $(2a) \in \mathbb{Z}$ with $(2b) \in \mathbb{Z}$. The identities derived for these ${}_2F_1$ are listed in appendix 1.

Some ${}_3F_2$ may be reduced via identities for Gaussian hypergeometric functions given by Abramowitz and Stegun (1972). In particular

$$
\begin{aligned}
{}_3F_2\left[\begin{matrix} a, b, b+\frac{1}{2} \\ a+1, 2b \end{matrix}; z\right] &= a \int_0^1 x^{a-1}{}_2F_1\left[\begin{matrix} b, b+\frac{1}{2} \\ 2b \end{matrix}; zx\right] \mathrm{d}x \\
&= \frac{a4^b}{z^a} \int_{(1-z)^{1/2}}^{1} (1-y)^{a-1}(1+y)^{a-2b}\, \mathrm{d}y
\end{aligned}
$$

and

$$
\begin{aligned}
{}_3F_2\left[\begin{matrix} a, b, b+\frac{1}{2} \\ a+1, \frac{1}{2} \end{matrix}; z\right] &= a \int_0^1 x^{a-1}{}_2F_1\left[\begin{matrix} b, b+\frac{1}{2} \\ \frac{1}{2} \end{matrix}; zx\right] \mathrm{d}x \\
&= \frac{a}{z^a} \int_0^{z^{1/2}} y^{2a-1}[(1+y)^{-2b} + (1-y)^{-2b}]\, \mathrm{d}y.
\end{aligned}
$$

It is possible to complete the integration for particular values of b. This should extend for general b but this has not been completed.

A further extension of these hypergeometric functions is the set of functions containing more than one pair of parameters which differ by one. The order of functions of this type can sometimes be lowered by applying the contiguous relation (Rainville 1960)

$$
\begin{aligned}
&{}_pF_q\left[\begin{matrix} a, b, \ldots \\ a+1, b+1, \ldots \end{matrix}; z\right] \\
&\quad = \frac{a}{(a-b)}\, {}_{p-1}F_{q-1}\left[\begin{matrix} b, \ldots \\ b+1, \ldots \end{matrix}; z\right] - \frac{b}{(a-b)}\, {}_{p-1}F_{q-1}\left[\begin{matrix} a, \ldots \\ a+1, \ldots \end{matrix}; z\right].
\end{aligned} \tag{8}
$$

When $a = b$ an alternative procedure is required.

Consider

$$
H = {}_{p+1}F_p\left[\begin{matrix} a, \ldots, a, b \\ a+1, \ldots, a+1 \end{matrix}; z\right].
$$

Laplace transformation yields

$$
H = -\frac{(-a)^p}{\Gamma(p)} \int_0^1 x^{a-1}(1-zx)^{-b} \ln^{(p-1)} x\, \mathrm{d}x.
$$

This may be integrated by parts. Defining the indefinite integral

$$v_i = \int \frac{1}{x} \dots \int \frac{1}{x} \int \frac{x^{a-1}}{(1-zx)} \mathrm{d}x \dots \mathrm{d}x$$

where the subscript i implies there are i integrations, gives

$$H = \lim_{\varepsilon \to 0} (-a)^p \left((-1)^p \int_\varepsilon^1 x^{-1} v_{p-1} \, \mathrm{d}x + \left[\sum_{i=1}^{p-1} v_i \ln^{(p-i)} x \frac{(-1)^i}{\Gamma(p+1-i)} \right]_\varepsilon^1 \right). \tag{9}$$

The results for the function

$${}_3F_2\left[\begin{matrix} a, a, b \\ a+1, a+1 \end{matrix}; z\right]$$

with $a \in \mathbb{Z}^+$ and $(2b) \in \mathbb{Z}$ are reported in appendix 1. A special case of this has been described by Inayat-Hussain (1987) in the course of studying critical-point behaviour of the n-vector model of phase transitions. It may also be possible to reduce other functions by the method of (9) but this has not yet been verified.

3.3. Identities obtained by l'Hôpital's rule

An alternative procedure applicable to some hypergeometric functions containing identical numerator parameters takes the limit $b \to a$ in (8) using l'Hôpital's rule (Luke 1969). This relies on a reduction of the hypergeometric functions in the right-hand side of (8) valid for all a and b. For example, the contiguous relations and Saalschutz's theorem yield

$${}_3F_2\left[\begin{matrix} a, c, -n \\ a+1, c-1-n \end{matrix}; 1\right] = \frac{\Gamma(n+1)(-1)^n \Gamma(c-1-n)}{\Gamma(c)} \left(a - \frac{(a+1-c)_{n+1}}{(a+1)_n} \right)$$

where $n \in \mathbb{Z}^+$. Using this expression, (8) and taking $b \to a$ an identity for a ${}_4F_3(1)$, which arises in the potential expansion in spherical polar coordinates, is obtained:

$$\begin{aligned} {}_4F_3&\left[\begin{matrix} a, a, c, -n \\ a+1, a+1, c-1-n \end{matrix}; 1\right] \\ &= \frac{\Gamma(n+1)(-1)^n \Gamma(c-1-n) a \Gamma(a+2+n-c) \Gamma(a+1)}{\Gamma(c) \Gamma(a+1+n) \Gamma(a+1-c)} \\ &\quad \times [\psi(a) + \psi(a+2+n-c) - \psi(a+1+n) - \psi(a+1-c)] \end{aligned} \tag{10}$$

where $n \in \mathbb{Z}^+$. This method of deriving (10) from the ${}_3F_2$ formula may be applied to extend the result to higher-order hypergeometric functions.

3.4. Reduction of series by rearrangement

The coefficients generated for the expansion of the few-particle wavefunction contain series which cannot be reduced using known transformations for hypergeometric functions. This is expected after surveying the reduction formulae listed for hypergeometrics. Focusing on single variable hypergeometric functions of argument ± 1, most reduction formulae express these as ratios of gamma functions, with identities (2) as the principal exceptions. The only other reductions are achieved by using l'Hôpital's rule relating ratios of gamma functions to polygamma functions (for example (10)). Although equations (5) and (7) were derived by a different procedure it is possible to use l'Hôpital's rule instead.

This highlights the deficiency of formulae expressing hypergeometric functions in terms of elementary functions other than gamma functions. Many terminating hypergeometric series of argument ± 1 cannot be expressed solely in terms of gamma functions, and would normally be called irreducible series. Some can be written in terms of well known irreducible series such as the truncated harmonic series, which may be expressed in terms of polygamma functions. It is therefore desirable to catalogue the reduction of hypergeometrics to polygamma functions more completely.

If the helium wavefunction is expanded as a series of Legendre polynomials the interelectron potential yields sums over 3-j symbols and expansion coefficients. Many of these are Saalschutzian hypergeometrics of unit argument and cannot be simplified by Saalschutz's theorem or the transformations relating well poised and Saalschutzian functions (Whipple 1926, 1936). The difficulty of solving Schrödinger's equation with the interelectron potential is thus related to the lack of reductions for Saalschutzian series. The principal result (Saalschutz's theorem) reduces a terminating ${}_3F_2(1)$ to a ratio of gamma functions. Abiodun (1980) simplified a restricted Saalschutzian function of arbitrary order, also to a ratio of gamma functions.

Consider

$$\sum_{p=1,2}^{n}\binom{n}{p}\frac{\Gamma(\frac{1}{2}+p/2)\Gamma(a+p/2)}{\Gamma(b-n/2+p/2)\Gamma(b-n/2+p/2+\frac{1}{2})}$$
$$=\frac{\Gamma(a+\frac{1}{2})n2^{2b-n}}{\Gamma(2b+1-n)\pi^{1/2}}\,{}_4F_3\left[\begin{matrix}1, a+\frac{1}{2}, 1-n/2, \frac{1}{2}-n/2\\ \frac{1}{2}+b-n/2, 1+b-n/2, \frac{3}{2}\end{matrix}; 1\right] \qquad (11)$$

where the notation $p=i,2$ means the sum over p starts at $p=i$ increasing in steps of 2. This occurs in the expansion of the helium wavefunction in polar coordinates (Gottschalk 1987). Equation (11) is reduced by applying a binomial identity to the series such that its length, initially $n/2$ for even n and $(n-1)/2$ for odd n, is reduced by one.

In this work rearrangement is often required for series containing a binomial term, of the form

$$\sum_{p=0}^{n}\binom{n}{p}f(p)$$

where f is usually a ratio of gamma functions. Since the binomial vanishes for $p>n$ the series limit is the upper parameter of the binomial.

In this work a binomial identity which reduces the top parameter of the binomial coefficient is applied, reducing the upper summation limit. An example is

$$\binom{n}{p}=\binom{n-2}{p}+2\binom{n-2}{p-1}+\binom{n-2}{p-2}$$

with (12)

$$\binom{n-2}{p-1}=\frac{(n-p)}{(p-1)}\binom{n-2}{p-2} \qquad \text{for } p\neq 1.$$

The $p=1$ term is treated separately. Repeated application of the identity reduces the original sum to a single term. When the binomial identity is applied, the complexity of the series increases if each term cannot be reduced to its original size, i.e. each application of the identity would yield series of decreasing length but with terms of increasing complexity. The original series eventually reduces to another finite series. This is useful only if the second series is simpler than the original.

Applying (12) k times to the series with p odd, (11) yields

$$\sum_{p=1,2}^{n}\binom{n}{p}\frac{\Gamma(\frac{1}{2}+p/2)\Gamma(a+p/2)}{\Gamma(b+p/2-n/2)\Gamma(b+p/2-n/2+\frac{1}{2})}$$

$$=\sum_{p=1,2}^{n-2k}\binom{n-2k}{p}\frac{\Gamma(\frac{1}{2}+p/2)\Gamma(a+p/2)Q(k,p)}{\Gamma(b+p/2-n/2+k)\Gamma(b+p/2-n/2+\frac{1}{2}+k)}$$

$$+2\Gamma(a+\tfrac{1}{2})\sum_{r=0}^{k-1}\frac{Q(r,1)}{\Gamma(b-n/2+\frac{1}{2}+r)\Gamma(b-n/2+1+r)} \tag{13}$$

where

$$Q(k,p)=(b-n/2+p/2+k-1)(b-n/2+p/2+k-\tfrac{1}{2})Q(k-1,p)$$

$$-(p+2k-n)(a+p/2)Q(k-1,p+2)$$

$$+\tfrac{1}{2}(p+1)(a+p/2)Q(k-1,p+2) \qquad k>0$$

and $Q(0,p)=1$.

By evaluating $Q(k,p)$ for various k it is seen that $Q(k,p)$ is independent of p if $b=1/2+a/2$ and is linear in p if $b=1+a/2$. It can be proved by induction that

$$Q(k,p)=\frac{\Gamma(n+a)}{\Gamma(n+a-2k)4^k} \qquad \text{for } b=\tfrac{1}{2}+a/2$$

and

$$Q(k,p)=\frac{\Gamma(n+a-1)}{\Gamma(n+a-2k+1)4^k}[(n+a-2k)(n+a-2k-1)$$

$$+k(2p+4a)] \qquad \text{for } b=1+a/2.$$

The final expression depends on whether n is even or odd. For odd n (13) yields

$$\sum_{p=1,2}^{n}\binom{n}{p}\frac{\Gamma(\frac{1}{2}+p/2)\Gamma(a+p/2)}{\Gamma(b+p/2-n/2)\Gamma(b+p/2-n/2+\frac{1}{2})}$$

$$=\frac{\Gamma(a+\frac{1}{2})Q(n/2-\frac{1}{2},1)}{\Gamma(b)\Gamma(b+\frac{1}{2})}$$

$$+2\Gamma(a+\tfrac{1}{2})\sum_{r=0}^{n/2-3/2}\frac{Q(r,1)}{\Gamma(b-n/2+\frac{1}{2}+r)\Gamma(b-n/2+1+r)}$$

while for even n the series equals

$$\frac{2\Gamma(a+\frac{1}{2})Q(n/2-1,1)}{\Gamma(b-\frac{1}{2})\Gamma(b)}+2\Gamma(a+\tfrac{1}{2})\sum_{r=0}^{n/2-2}\frac{Q(r,1)}{\Gamma(b-n/2+\frac{1}{2}+r)\Gamma(b-n/2+1+r)}.$$

These results are useful because the series on the right of these equations, although of the same length as the orignal, are less complex.

For $2a$ an odd integer the expressions for n odd and n even can be unified by expanding the ratios of gamma functions by partial fraction decomposition (Rainville 1960, pp 82–3), noting that

$$\psi(n+z)-\psi(z)=\sum_{p=0}^{n-1}(p+z)^{-1}.$$

This eliminates the sums which terminate at a point depending on n, producing sums whose upper limits are functions of a.

The remainder of the reduction applies finite series rearrangement and properties of the digamma functions. The lengthy calculation is omitted here. Details are described by Gottschalk (1987).

The unified results are

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix}1, a, \frac{1}{2}-n/2, -n/2\\ 1/4+a/2-n/2, \frac{3}{4}+a/2-n/2, 3/2\end{matrix}; 1\right]\\
&\quad=\frac{\Gamma(n+a+\frac{1}{2})}{(n+1)\Gamma(n-a+\frac{1}{2})}\left\{\frac{(-1)^{n+a}\pi 4^{a-1}}{\Gamma(2a)}+\sum_{j=1}^{a}\frac{(-1)^j}{\Gamma(2a+1-j)\Gamma(j)}\right.\\
&\quad\quad\left.\times\left[\psi\left(\frac{a}{2}-\frac{j}{2}+\frac{n}{2}+\frac{5}{4}\right)-\psi\left(\frac{j}{2}+\frac{n}{2}-\frac{a}{2}+\frac{3}{4}\right)\right]\right\}
\end{aligned}
\tag{14}
$$

and

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix}1, a, \frac{1}{2}-n/2, -n/2\\ \frac{3}{4}+a/2-n/2, \frac{5}{4}+a/2-n/2, \frac{3}{2}\end{matrix}; 1\right]\\
&\quad=\frac{\Gamma(n+a-\frac{1}{2})}{(n+1)\Gamma(n-a-\frac{1}{2})}\left\{-\frac{4^{a-1}(-1)^{a+n}\pi}{\Gamma(2a)}-\frac{\pi(-1)^{a+n}}{\Gamma(a+\frac{1}{2})^2}\right.\\
&\quad\quad+\sum_{j=1}^{a}\frac{(-1)^j}{\Gamma(j)\Gamma(2a+1-j)}\left[\psi\left(\frac{a}{2}-\frac{j}{2}+\frac{n}{2}+\frac{3}{4}\right)-\psi\left(\frac{j}{2}-\frac{a}{2}+\frac{n}{2}+\frac{1}{4}\right)\right]\\
&\quad\quad\left.-2a\sum_{j=1}^{2a+2}\frac{(n+a+\frac{3}{2}-j)(-1)^j}{\Gamma(j)\Gamma(2a+3-j)}\left[(-1)^{n+1}\psi\left(\frac{a}{2}+\frac{5}{4}-\frac{j}{2}\right)-\psi\left(\frac{a}{2}+\frac{n}{2}-\frac{j}{2}+\frac{7}{4}\right)\right]\right\}.
\end{aligned}
\tag{15}
$$

For convenience the earlier definitions of the parameters a and n have been shifted by $\frac{1}{2}$ and 1 respectively. The conditions on the new a and n is that both are positive integers. While digamma functions, $\psi(z)$, and π have infinite series representations, they combine in (14) and (15) to form finite series. The presentation used here was found to be the most useful. Formulae (14) and (15) are useful because, in the potential expansions, a is always a number rather than a variable.

Reduction formulae have not been derived for other functions of interest, such as

$$
{}_5F_4\left[\begin{matrix}a, a, \alpha, -n/2, \frac{1}{2}-n/2\\ a+1, a+1, \frac{1}{2}+\alpha/2-n/2, 1+\alpha/2-n/2\end{matrix}; 1\right] \qquad \text{where } a, n\in\mathbb{Z}^{+}. \tag{16}
$$

This is similar to

$$
{}_4F_3\left[\begin{matrix}a, \alpha, -n/2, \frac{1}{2}-n/2\\ a+1, \frac{1}{2}+\alpha/2-n/2, 1+\alpha/2-n/2\end{matrix}; 1\right]
$$

which reduces by transforming to an equivalent ${}_4F_3$ (Whipple 1926), expressing this as a finite series of ${}_3F_2$ reducible via Saalschutz's theorem, then applying standard identites (Slater 1966) to simplify the result. Although the even and odd n functions are considered separately the results can be united. The reduction of (16) does not follow using this with l'Hôpital's rule (§ 3.3) because the ${}_4F_3$ formula contains a sum terminating at a, and is only valid for integral a.

4. Reduction of order of hypergeometric functions

Hypergeometric functions arising from the few-body Schrödinger equation are rarely reducible by direct application of the identities in §§ 2 and 3. It is more common for the functions to be of higher order, i.e. to contain pairs of parameters such as $\{a, a+m\}$, $m \in \mathbb{Z}$ rather than $\{a, a+1\}$. These may be related to the core functions defined in §§ 2 and 3 by repeated application of the contiguous relations (Rainville 1960). Note that Rainville (1945) listed an extra contiguous relation which is sometimes useful, although it is not independent of those given by Rainville (1960).

Formulae equivalent to applying contiguous relations k times are presented in appendix 2. Their proof (by induction) is not described here.

The transformation formulae listed in appendix 2 are helpful in reducing the order of quite general hypergeometric functions. Many functions arising in this work may be transformed more efficiently using the more restrictive formulae given by Karlsson (1971, 1974). Minton (1970) gives a relation for a function with the property that some numerator parameters are greater by a positive integer than denominator parameters. This is generalised by Karlsson by relating the function to a sum of hypergeometric functions of lower order;

$$
\begin{aligned}
&{}_pF_q\left[\begin{matrix} b_1+m_1, \ldots, b_n+m_n, a_{n+1}, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}; z\right] \\
&\quad = \sum_{j_1=0}^{m_1} \cdots \sum_{j_n=0}^{m_n} A(j_1, \ldots, j_n) z^{J_n} {}_{p-n}F_{q-n}\left[\begin{matrix} a_{n+1}+J_n, \ldots, a_p+J_n \\ b_{n+1}+J_n, \ldots, b_q+J_n \end{matrix}; z\right]
\end{aligned} \tag{17}
$$

where $J_n = j_1 + \ldots + j_n$ and

$$
A(j_1, \ldots, j_n) = \binom{m_1}{j_1} \cdots \binom{m_n}{j_n} \frac{(b_2+m_2)_{J_1}(b_3+m_3)_{J_2} \ldots (b_n+m_n)_{J_{n-1}}(a_{n+1})_{J_n} \ldots (a_p)_{J_n}}{(b_1)_{J_1}(b_2)_{J_2} \ldots (b_n)_{J_n}(b_{n+1})_{J_n} \ldots (b_q)_{J_n}}.
$$

Srivastava (1973) provides a simpler proof. Karlsson (1974) also related a hypergeometric function, which has some numerator parameters smaller than some denominator parameters, to functions of lower order. His least restrictive result is

$$
\begin{aligned}
&{}_pF_q\left[\begin{matrix} a_1, \ldots, a_n, a_{n+1}, \ldots, a_p \\ a_1+m_1+1, \ldots, a_n+m_n+1, b_{n+1}, \ldots, b_q \end{matrix}; z\right] \prod_{r=1}^{n} \frac{m_r!}{(a_r)_{m_r+1}} \\
&\quad = \sum_{j_1=0}^{m_1} \cdots \sum_{j_n=0}^{m_n} \left(\prod_{r=1}^{n} \frac{(-m_r)_{j_r}}{j_r!(a_r+j_r)}\right) {}_pF_q\left[\begin{matrix} a_1+j_1, \ldots, a_n+j_n, a_{n+1}, \ldots, a_p \\ a_1+j_1+1, \ldots, a_n+j_n+1, b_{n+1}, \ldots, b_q \end{matrix}; z\right].
\end{aligned} \tag{18}
$$

Assuming (i) $\{a_1, \ldots, a_n\}$ are all different and (ii) if $a_r - a_i = N \in \mathbb{N}$ then $N > m_i$ is satisfied, the required reduction of order is achieved:

$$
{}_pF_q\left[\begin{matrix} a_1, \ldots, a_n, a_{n+1}, \ldots, a_p \\ a_1+m_1+1, \ldots, a_n+m_n+1, b_{n+1}, \ldots, b_q \end{matrix}; z\right] \prod_{r=1}^{n} \frac{m_r!}{(a_r)_{m_r+1}}
$$
$$
= \sum_{i=1}^{n} \sum_{j_1=0}^{m_1} \cdots \sum_{j_n=0}^{m_n} \frac{1}{(a_i+j_i)} \left(\prod_{s=1}^{n} \frac{(-m_s)_{j_s}}{j_s!} \right)
$$
$$
\times \left(\prod_{r=1, r\neq i}^{n} \frac{1}{(a_r+j_r-a_i-j_i)} \right) {}_{p-n+1}F_{q-n+1}\left[\begin{matrix} a_i+j_i, a_{n+1}, \ldots, a_p \\ a_i+j_i+1, b_{n+1}, \ldots, b_q \end{matrix}; z\right]. \tag{19}
$$

Karlsson did not realise that the multiple summation can be greatly reduced using

$$
\sum_{j_r=0}^{m_r} \frac{(-m_r)_{j_r}}{j_r!(a_r+j_r-a_i-j_i)} = \frac{\Gamma(a_r-a_i-j_i)m_r!}{\Gamma(a_r+m_r-a_i-j_i+1)}
$$

(proven by Vandermonde's theorem), to give the simpler form of (19):

$$
{}_pF_q\left[\begin{matrix} a_1, \ldots, a_n, a_{n+1}, \ldots, a_p \\ a_1+m_1+1, \ldots, a_n+m_n+1, b_{n+1}, \ldots, b_q \end{matrix}; z\right] \prod_{r=1}^{n} \frac{1}{(a_r)_{m_r+1}}
$$
$$
= \sum_{i=1}^{n} \sum_{j=0}^{m_i} \frac{(-m_i)_j}{j!(a_i+j)m_i!} \left(\prod_{r=1, r\neq i}^{n} \frac{1}{(a_r-a_i-j)_{m_r+1}} \right)
$$
$$
\times {}_{p-n+1}F_{q-n+1}\left[\begin{matrix} a_i+j, a_{n+1}, \ldots, a_p \\ a_i+j+1, b_{n+1}, \ldots, b_q \end{matrix}; z\right]. \tag{20}
$$

Most hypergeometric series with integral parameter differences encountered in spherical polar expressions for few-body wavefunctions are initially reduced using (17) to functions of the type

$$
{}_{p+1}F_p\left[\begin{matrix} a_1, \ldots, a_p, b \\ a_1+m_1+1, \ldots, a_p+m_p+1 \end{matrix}; z\right] \qquad m_i \in \mathbb{Z}^+. \tag{21}
$$

The procedure for reducing functions of this form depends on whether b is non-integral, a positive integer or a negative integer.

When the conditions on the a_i are satisfied (21) may be reduced to a finite sum of ${}_2F_1$ using transformation (20), resulting in

$$
{}_{p+1}F_p\left[\begin{matrix} a_1, \ldots, a_p, b \\ a_1+m_1+1, \ldots, a_p+m_p+1 \end{matrix}; z\right] \prod_{r=1}^{p} \frac{\Gamma(a_r)}{\Gamma(a_r+m_r+1)}
$$
$$
= \sum_{i=1}^{p} \sum_{j=0}^{m_i} \frac{(-m_i)_j}{j!\,m_i!(a_i+j)} \prod_{r=1, r\neq i}^{p} \frac{\Gamma(a_r-a_i-j)}{\Gamma(a_r+m_r-a_i-j+1)}
$$
$$
\times {}_2F_1\left[\begin{matrix} a_i+j, b \\ a_i+j+1 \end{matrix}; z\right]. \tag{22}
$$

If some of the a_i are identical the more general transformation (18) is required. In many important cases this restriction is satisfied, so it is worthwhile to examine (22) in more detail. Series of this type encountered in this work often have $z = 1$. Equation

(22) does not seem useful at first since the hypergeometric functions ${}_2F_1(a_i+j, b; a_i+j+1; 1)$ are convergent only for $\mathrm{Re}(1-b)>0$. In general the functions encountered have $\mathrm{Re}(1-b)<0$. However, this problem is overcome by the following procedure.

The derivation for the formula for $z=1$ is simplified by preferring Karlsson's equation (19) to the reduced version (20). With $a_{n+1}=b$, $p=n+1$ and $q=n$ the hypergeometric on the right-hand side of (19) becomes a ${}_2F_1(z)$ which may be written as a sum of two functions using the linear relations (Abramowitz and Stegun 1972):

$$ {}_2F_1\left[\begin{matrix} a, b \\ a+1 \end{matrix}; z\right] = \frac{\Gamma(a+1)\Gamma(1-b)}{\Gamma(a+1-b)}\, {}_1F_0\left[\begin{matrix} a \\ - \end{matrix}; 1-z\right] + \frac{(1-z)^{1-b}a}{(b-1)}\, {}_2F_1\left[\begin{matrix} 1, a+1-b \\ 2+b \end{matrix}; 1-z\right]. $$

Taking the limit of $z\to 1$ in (19) yields

$$ \begin{aligned} {}_{p+1}F_p&\left[\begin{matrix} a_1,\ldots,a_p,b \\ a_1+m_1+1,\ldots,a_p+m_p+1 \end{matrix}; 1\right] \prod_{r=1}^{p} \frac{m_r!}{(a_r)_{m_r+1}} = \Gamma(1-b)\sum_{i=1}^{p}\sum_{j_1=0}^{m_1}\cdots\sum_{j_p=0}^{m_p} \\ &\times \frac{\Gamma(a_i+j_i)}{\Gamma(a_i+j_i+1-b)}\left(\prod_{s=1}^{p}\frac{(-m_s)_{j_s}}{j_s!}\right)\left(\prod_{r=1,r\neq i}^{p}\frac{1}{(a_r+j_r-a_i-j_i)}\right) \\ &+\lim_{z\to 1}\frac{1}{(b-1)}\sum_{j_1=0}^{m_1}\cdots\sum_{j_p=0}^{m_p}\left(\prod_{s=1}^{p}\frac{(-m_s)_{j_s}}{j_s!}\right) \\ &\times\left(\sum_{i=1}^{p}\prod_{r=1,r\neq i}^{p}\frac{1}{(a_r+j_r-a_i-j_i)}\right)(1-z)^{1-b}. \end{aligned} $$

It can be shown by induction over p that

$$ \sum_{i=1}^{p}\prod_{r=1,r\neq i}^{p}(a_r+j_r-a_i-j_i)^{-1}=0 $$

so the last part vanishes for all values of z. Vandermonde's theorem again enables most of the remaining sums to be evaluated, resulting in

$$ \begin{aligned} {}_{p+1}F_p&\left[\begin{matrix} a_1,\ldots,a_p,b \\ a_1+m_1+1,\ldots,a_p+m_p+1 \end{matrix}; 1\right] \\ &=\Gamma(1-b)\left(\prod_{r=1}^{p}\frac{\Gamma(a_r+m_r+1)}{\Gamma(a_r)}\right)\sum_{i=1}^{p}\sum_{j=0}^{m_i}\frac{(-1)^j\Gamma(a_i+j)}{\Gamma(1-j+m_i)j!\Gamma(a_i+j+1-b)} \\ &\times\prod_{r=1,r\neq i}^{p}\frac{\Gamma(a_r-a_i-j)}{\Gamma(a_r+m_r-a_i-j+1)} \qquad m_i\in\mathbb{Z}^+;\ b\notin\mathbb{Z}. \end{aligned} \tag{23} $$

This is equivalent to applying Gauss's theorem to the ${}_2F_1$ in (22) without regard to the validity of the operation.

When b is a negative integer, $z=1$ and the restrictions on a_i are satisfied, the ${}_2F_1$ in (22) are terminating series, which can be evaluated immediately using Vandermonde's theorem. The final expression is identical to (23).

If b is a positive integer the hypergeometric function must be related to (2), (3) or one of the series in appendix 1. This requires the hypergeometric (21) to be transformed, so that b is replaced by 1. Note that (21) equals

$$ {}_{p+2}F_{p+1}\left[\begin{matrix} a_1, \ldots, a_p, b, 1 \\ a_1+m_1+1, \ldots, a_p+m_p+1, 1 \end{matrix}; z\right]. \tag{24} $$

Karlsson's transformation (17) is not helpful here since b may be eliminated, but the 1 in the numerator will also be increased. The transformation required is one of the generalised contiguous relations (equation (A2.4)). This transformation increases the 1 in the denominator parameters to cancel with the b in the numerator of (24). Applying (18) or (20) to the resulting hypergeometric functions enables complete reduction via the identities in §§ 2 and 3. A particular case that arises is $b=1$, $z=1$ when the restrictions on a_i in (20) and (22) are satisfied. The transformation (22) does not apply as the ${}_2F_1$ are divergent. Instead one obtains from (20) the result

$$ {}_{p+1}F_p\left[\begin{matrix} a_1, \ldots, a_p, 1 \\ a_1+m_1+1, \ldots, a_p+m_p+1 \end{matrix}; 1\right] = \left(\prod_{r=1}^{p} \frac{\Gamma(a_r+m_r+1)}{\Gamma(a_r)}\right) $$
$$ \times \sum_{i=1}^{p-1} \sum_{j=0}^{m_i} \sum_{k=0}^{m_p} \frac{(-1)^{j+k}(\psi(a_i+j)-\psi(a_p+k))}{j!\,k!\,\Gamma(1-j+m_i)\Gamma(1-k+m_p)(a_i+j-a_p-k)} $$
$$ \times \prod_{r=1, r\neq i}^{p-1} \frac{\Gamma(a_r-a_i-j)}{\Gamma(a_r+m_r-a_i-j+1)}. $$

5. Conclusion

The reduction of a number of generalised hypergeometric functions of one variable has been described. The simplification of a function containing numerator and denominator parameters differing by integers proceeds via the reduction in the function's order, to a sum of hypergeometrics. These may be reduced via a collection of identities for low-order functions. New identities belonging to this set have been derived, and known results have been generalised.

The derivation of reduction formulae for hypergeometric functions is less systematic than the similar problem of integration. Methods applied in this work have been classified so that the identities here may be extended.

Most recent work concerns hypergeometric functions of intrinsically high order (i.e. they are not reducible in order using the transformations in § 4) or functions of two or more variables, whereas the results in this article relate to functions of low order with some parameters differing by integers. These have been chosen deliberately because they arise in particular calculations, namely the solution of the Schrödinger equation by series expansion.

Methods used in this work have also been applied to simple multiple series, expressible as multivariable hypergeometric functions. Preliminary results have been described by Gottschalk (1987).

Acknowledgments

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(1966), have been programmed as external files for that package. The pattern matching feature, and the ease of encoding identities containing a number of summations which is unspecified until the application of the identity, makes packages such as SMP extremely useful in this work.

Appendix 1. Reduction formulae for some hypergeometric functions

$$
{}_2F_1\left[\begin{matrix} a, b \\ a+1 \end{matrix}; z\right] = -\frac{\Gamma(a+1)\Gamma(1-b)}{\Gamma(a+1-b)z^a}[(1-z)^{1-b}-1]
$$

$$
-\frac{\Gamma(a+1)(1-z)^{1-b}}{\Gamma(a+1-b)z^a}\sum_{i=1}^{a-1}\frac{\Gamma(i+1-b)z^i}{\Gamma(i+1)} \qquad a \in \mathbb{Z}^+.
$$

$$
{}_2F_1\left[\begin{matrix} a, b \\ a+1 \end{matrix}; z\right] = -\frac{(1-z)^{1-b}\Gamma(a+1)}{z^{a+1/2}\Gamma(a+1-b)}\mathop{\mathsf{S}}_{i=1}^{a-1/2}\frac{\Gamma(i+\frac{1}{2}-b)z^i}{\Gamma(i+\frac{1}{2})}
$$

$$
+\frac{\Gamma(a+1)\Gamma(\frac{3}{2}-b)\Gamma(b-\frac{1}{2})}{\Gamma(a+1-b)\Gamma(b)z^{a-1/2}\pi^{1/2}}\left(\sum_{i=1}^{b-1}\frac{\Gamma(i)}{(1-z)^i\Gamma(i+\frac{1}{2})}+\frac{1}{\sqrt{|z|}}\omega(z)\right)
$$

where $(2a) \in \mathbb{Z}$, $a \notin \mathbb{Z}$, $b \in \mathbb{Z}^+$, and

$$
\omega(z) = \begin{cases} 2\tan^{-1}\sqrt{-z} & z<0 \\ \ln(1+\sqrt{z})-\ln(1-\sqrt{z}) & z>0. \end{cases}
$$

$$
{}_2F_1\left[\begin{matrix} a, b \\ a+1 \end{matrix}; z\right] = -\frac{(1-z)^{1-b}\Gamma(a+1)}{z^{a+1/2}\Gamma(a+1-b)}\mathop{\mathsf{S}}_{i=1}^{a-1/2}\frac{\Gamma(i+\frac{1}{2}-b)z^i}{\Gamma(i+\frac{1}{2})}+\frac{\Gamma(a+1)\Gamma(\frac{3}{2}-b)}{\Gamma(a+1-b)z^{a-1/2}\pi^{1/2}}
$$

$$
\times\left(\mathop{\mathsf{S}}_{i=1}^{b-1/2}\frac{(1-z)^{1/2-i}\Gamma(b-\frac{1}{2})\Gamma(i-\frac{1}{2})}{\Gamma(i)\Gamma(b)}+\frac{2\Gamma(1-b)}{\Gamma(\frac{3}{2}-b)\pi^{1/2}\sqrt{|z|}}\varphi(z)\right)
$$

where $(2a) \in \mathbb{Z}$, $a \notin \mathbb{Z}$, $(2b) \in \mathbb{Z}$, $b \notin \mathbb{Z}$ and not both $a > \frac{1}{2}$ and $b > \frac{1}{2}$, and

$$
\varphi(z) = \begin{cases} \ln(\sqrt{1-z}+\sqrt{-z}) & z<0 \\ \tan^{-1}\sqrt{z/(1-z)} & z>0 \end{cases}.
$$

$$
{}_2F_1\left[\begin{matrix} a, b \\ a+1 \end{matrix}; z\right] = \frac{a\Gamma(b-a)}{\Gamma(b)}\sum_{i=1}^{b-1/2}\frac{\Gamma(i-\frac{1}{2})}{(1-z)^{i-1/2}\Gamma(i+\frac{1}{2}-a)}-\frac{\Gamma(a+1)\Gamma(1-b)}{\Gamma(a+1-b)z^{a-1/2}\pi^{1/2}}
$$

$$
\times\left(\sqrt{1-z}\sum_{i=1}^{a-1/2}\frac{\Gamma(i)z^{i-1}}{\Gamma(i+\frac{1}{2})}-\frac{2}{\pi^{1/2}\sqrt{|z|}}\varphi(z)\right)
$$

where $(2a) \in \mathbb{Z}^+$, $a \notin \mathbb{Z}^+$, $(2b) \in \mathbb{Z}^+$, $b \notin \mathbb{Z}^+$.

$$
{}_3F_2\left[\begin{matrix} a, a, b \\ a+1, a+1 \end{matrix}; z\right] = \frac{\Gamma(a+1)a\Gamma(2-b)[(1-z)^{2-b}-1]}{\Gamma(a+1-b)z^a}\sum_{i=1}^{a-1}\frac{1}{i(i+1-b)}
$$

$$
+\frac{a\Gamma(a+1)(1-z)^{2-b}}{\Gamma(a+1-b)z^a}\sum_{i=1}^{a-1}\frac{1}{i(i+1-b)}\sum_{j=1}^{i-1}\frac{\Gamma(j+2-b)z^j}{\Gamma(j+1)}
$$

$$
-\frac{a\Gamma(a+1)\Gamma(1-b)}{\Gamma(a+1-b)z^a}\vartheta(b, z)
$$

where

$$\vartheta(b,z)=\begin{cases}\displaystyle\sum_{j=1}^{b-2}\frac{[(1-z)^{-j}-1]}{j}-\ln(1-z) & a\in\mathbb{Z}^+,\ b\in\mathbb{Z}^+,\ b\geqslant a+1\\ \displaystyle\mathop{\mathrm{S}}_{j=1}^{b-3/2}\frac{[(1-z)^{1/2-j}-1]}{(j-\frac{1}{2})}+2\ln|1-\sqrt{1-z}|-2\ln|z|+2\ln 2 & \\ & a\in\mathbb{Z}^+,\ (2b)\in\mathbb{Z},\ b\notin\mathbb{Z}.\end{cases}$$

Appendix 2. Transformations derived from the contiguous relations

In the following, restrictions on a_i and b_j are such that the right-hand sides are not singular, and we use the notation:

$$F={}_pF_q\left[\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix};z\right]$$

$$F(a_i+h)={}_pF_q\left[\begin{matrix}a_1,\ldots,a_i+h,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix};z\right]$$

$$F(b_i+h)={}_pF_q\left[\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_i+h,\ldots,b_q\end{matrix};z\right].$$

$(a_i-b_j+1)F=a_iF(a_i+1)-(b_j-1)F(b_j-1)\Rightarrow$

$$F=\frac{(a_i)_k}{(1+a_i-b_j)_k}\sum_{n=0}^{k}\binom{k}{n}\frac{(-1)^n(1-b_j)_n}{(1-a_i-k)_n}F(a_i+k-n,b_j-n). \tag{A2.1}$$

$(a_i-a_j)F=a_iF(a_i+1)-a_jF(a_j+1)\Rightarrow$

$$F=-\sum_{n=0}^{k}\binom{k}{n}\frac{(a_j)_n(1-a_i+a_j)_{n-k-1}(a_i-a_j+k-2n)}{(1-a_i)_{n-k}(1-a_i+a_j)_n}\times F(a_i+k-n,a_j+n)\qquad k<|a_i-a_j|. \tag{A2.2}$$

$(a_i-a_j)F=a_iF(a_i+1)-a_jF(a_j+1)\Rightarrow$

$$F=\frac{(a_i)_k}{(a_i-a_j)_k}F(a_i+k)-a_j\sum_{n=1}^{k}\frac{(a_i)_{n-1}}{(a_i-a_j)_n}F(a_i+n-1,a_j+1). \tag{A2.3}$$

$b_jF=a_iF(a_i+1,b_j+1)+(b_j-a_i)F(b_j+1)\Rightarrow$

$$F=\frac{(a_i)_k}{(b_j)_k}\sum_{n=0}^{k}\binom{k}{n}\frac{(-1)^n(b_j-a_i)_n}{(1-a_i-k)_n}F(a_i+k-n,b_j+k). \tag{A2.4}$$

$(a_i-1)F=(a_i-a_j-1)F(a_i-1)+a_jF(a_i-1,a_j+1)\Rightarrow$

$$F=\frac{(1-a_i+a_j)_k}{(1-a_i)_k}\sum_{n=0}^{k}\binom{k}{n}\frac{(-1)^n(a_j)_n}{(1-a_i+a_j)_n}F(a_i-k,a_j+n). \tag{A2.5}$$

$(a_i-1)F=(a_i-a_j-1)F(a_i-1)+a_jF(a_i-1,a_j+1)\Rightarrow$

$$F=\frac{(1-a_i+a_j)_k}{(1-a_i)_k}F(a_i-k)-a_j\sum_{n=1}^{k}\frac{(1-a_i+a_j)_{n-1}}{(1-a_i)_n}F(a_i-n,a_j+1). \tag{A2.6}$$

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